MATH 20D Spring 2023 Lecture 26.

Solving Systems of Equations Using Eigenvectors.

Outline

Contents

- Let A be a 2-by-2 matrix. **Eigenvalues** of A are the scalars $\lambda \in \mathbb{C}$ such that

$$\det(A - \lambda I) = 0$$

where $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is the 2-by-2 identity matrix.

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$$(A - \sqrt{5}I)\mathbf{v} = \begin{pmatrix} 2 - \sqrt{5} & 1 \\ 1 & -2 - \sqrt{5} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \mathbf{0}.$$

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So $\mathbf{v} = \mathrm{col}(1, -(2 - \sqrt{5}))$ is an eigenvector with eigenvalue λ_1.

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For each of the matrices below. Determine whether A admits a pair of linearly independent eigenvectors.

(a) $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ **(b)** $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$

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- If $\mathbf{v}$ is an eigenvector of A with eigenvalue λ then any non-zero scalar multiple of $\mathbf{v}$ is an eigenvector of A with eigenvalue λ.So eigenvectors of A corresponding to **distinct** eigenvalues are always linearly independent.

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$$ay''(t) + by'(t) + cy(t) = 0 \tag{1}$$
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$$A = \begin{pmatrix} 0 & 1 \\ -c/a & -b/a \end{pmatrix}$$
- Indeed $a\lambda^2 + b\lambda + c\lambda = a \cdot \det(A - \lambda I)$ and the matrix A can be used to express (1) as the matrix equation
$$\mathbf{x}'(t) = A\mathbf{x}(t) \quad \text{where } \mathbf{x}(t) = \mathrm{col}(y(t), y'(t)).$$

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Definition

We say that two vector valued function $\mathbf{x}_1, \mathbf{x}_2 : \mathbb{R} \to \mathbb{C}^2$ are ***linearly dependent on*** $(-\infty, \infty)$ *if there exists a scalar $\alpha \in \mathbb{C}$ such that either*

$$\mathbf{x}_1(t) = \alpha\mathbf{x}_2(t) \quad \text{or} \quad \mathbf{x}_2(t) = \alpha \cdot \mathbf{x}_2(t)$$

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Suppose $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ are linearly independent solutions to (2).

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$$\mathbf{x}(t) = C_1\mathbf{x}_1(t) + C_2\mathbf{x}_2(t).$$

where C_1 and C_2 are constant.

- To prove the theorem on the previous slide one shows that for any initial value problem
$$\mathbf{x}'(t) = A\mathbf{x}(t), \qquad \mathbf{x}(0) = \mathbf{x}_0 \tag{3}$$
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- We have $\det(X(0)) = \mathrm{Wr}[\mathbf{x}_1, \mathbf{x}_2](0) \neq 0$ by a version of Abel's theorem.

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$$\mathbf{x}_1(t) = e^{r_1 t}\mathbf{v}_1 \quad \text{and} \quad \mathbf{x}_2(t) = e^{r_2 t}\mathbf{v}_2$$

give a pair of linearly independent solutions to (4).

Example

Write down a general solution to the equation

$$\mathbf{x}'(t) = \begin{pmatrix} 2 & -3 \\ 1 & -2 \end{pmatrix} \mathbf{x}(t)$$